section setdefs

/Reference manual/Extended toolkit

The ‘definition’ parts of this section are a formal specification of those parts of the mathematical tool-kit which do not depend on numbers. All objects are defined before they are used, and all are defined explicitly.

$$\text{section } \mathit{setdefs}$$

1. Sets

1.0.0.1. Name

$\neq$	–	Inequality
$\notin$	–	Non-membership

1.0.0.2. Definition

$$\text{relation}(\_ \neq \_)$$

$$\_ \neq \_[X] == \{x, y : X \mid \lnot\ (x = y)\}$$

$$\text{relation}(\_ \notin \_)$$

$$\_ \notin \_[X] == \{x : X;\ S : \mathbb{P}\, X \mid \lnot\ (x \in S)\}$$

1.0.0.3. Name

$\varnothing$	–	Empty set
$\subseteq$	–	Subset relation
$\subset$	–	Proper subset relation
$\mathbb{P}_1$	–	Non-empty subsets

1.0.0.4. Definition

$$\varnothing[X] == \{x : X \mid \mathit{false}\}$$

$$\mathrm{relation}(\_ \subseteq \_)$$

$$\_ \subseteq \_[X] == \{S, T : \mathbb{P}\, X \mid (\forall x : S \bullet x \in T)\}$$

$$\mathrm{relation}(\_ \subset \_)$$

$$\_ \subset \_[X] == \{S, T : \mathbb{P}\, X \mid S \subseteq T \land S \neq T\}$$

$$\mathbb{P}_1\, X == \{a : \mathbb{P}\, X \mid a \neq \varnothing\}$$

1.0.0.5. Name

$\cup$	–	Set union
$\cap$	–	Set intersection
$\setminus$	–	Set difference
$\ominus$	–	Set symmetric difference

1.0.0.6. Definition

$$\text{function } 30 \text{ leftassoc } (\_ \cup \_)$$

$$\_ \cup \_[X] == \lambda\, S, T : \mathbb{P}\, X \bullet \{x : X \mid x \in S \lor x \in T\}$$

$$\text{function } 40 \text{ leftassoc } (\_ \cap \_)$$

$$\_ \cap \_[X] == \lambda\, S, T : \mathbb{P}\, X \bullet \{x : X \mid x \in S \land x \in T\}$$

$$\text{function } 30 \text{ leftassoc } (\_ \setminus \_)$$

$$\_ \setminus \_[X] == \lambda\, S, T : \mathbb{P}\, X \bullet \{x : X \mid x \in S \land x \notin T\}$$

$$\text{function } 25 \text{ leftassoc } (\_ \ominus \_)$$

$$\_ \ominus \_[X] == \lambda\, S, T : \mathbb{P}\, X \bullet \{x : X \mid \lnot\, (x \in S \Leftrightarrow x \in T)\}$$

1.0.0.7. Description These four functions are the only possible non-degenerate functions between two sets using only the information that they are of the same type as each other and that the result is of that same common type. The four functions form a closed family, in that if we take any two starting sets of the same type, and form any expression using any number of occurrences of those two sets and these four functions, the result will be one which could be expressed by a single use of one of these functions.

1.0.0.8. Name $\bigcup$ – Generalized union
$\bigcap$ – Generalized intersection

$$\bigcup[X] == \lambda\, A : \mathbb{P}(\mathbb{P}\, X) \bullet \{x : X \mid (\exists\, S : A \bullet x \in S)\}$$

$$\bigcap[X] == \lambda\, A : \mathbb{P}(\mathbb{P}\, X) \bullet \{x : X \mid (\forall\, S : A \bullet x \in S)\}$$

1.0.0.9. Name $\mathbb{F}$ – Finite sets
$\mathbb{F}_1$ – Non-empty finite sets

$$\mathrm{generic}(\mathbb{F}\ \_)$$

$$\mathbb{F}\, X == \bigcap\{A : \mathbb{P}(\mathbb{P}\, X) \mid \varnothing \in A \land \forall\, a : A;\ x : X \bullet a \cup \{x\} \in A\}$$

$$\mathbb{F}_1\, X == \mathbb{F}\, X \setminus \{\varnothing\}$$

$$\mathrm{relation}(\mathit{finite}\ \_)$$

$$\mathit{finite}\ \_[X] == \mathbb{F}\, X$$

1.0.0.10. Description A subset S of X is finite if and only if it is a member of the set of finite subsets of X, $S \in \mathbb{F}\, X$. The finite subsets of X form the smallest set which contains the empty set and is closed under the action of adding single elements of X. The non-empty finite subsets of X form the smallest set which contains the singleton sets of X and is closed under the action of adding single elements of X. The sets in $\mathbb{F}_1\, X$ are the non-empty members of $\mathbb{F}\, X$.

1.0.0.11. Name id – Identity relation

1.0.0.12. Definition

$$\text{generic}(id\ \_)$$

$$id\ X == \lambda\, x : X \bullet x$$

2. Relations

2.0.0.13. Name

$\leftrightarrow$	–	Binary relations
$\mapsto$	–	Maplet
first, second	–	Projection functions for ordered pairs

2.0.0.14. Definition

$$\text{generic}\ 5\ \text{rightassoc}\ (\_ \leftrightarrow \_)$$

$$X \leftrightarrow Y == \mathbb{P}(X \times Y)$$

$$\text{function}\ 10\ \text{leftassoc}\ (\_ \mapsto \_)$$

$$\_ \mapsto \_[X, Y] == \lambda\, p : X \times Y \bullet p$$

$$\mathit{first}[X, Y] == \lambda\, p : X \times Y \bullet p.1$$

$$\mathit{second}[X, Y] == \lambda\, p : X \times Y \bullet p.2$$

2.0.0.15. Name $\mathit{dom}, \mathit{ran}$ – Domain and range of a relation

2.0.0.16. Definition

$$\mathit{dom}[X, Y] == \lambda\, R : X \leftrightarrow Y \bullet \{p : R \bullet p.1\}$$

$$\mathit{ran}[X, Y] == \lambda\, R : X \leftrightarrow Y \bullet \{p : R \bullet p.2\}$$

2.0.0.17. Name

$\lhd$ – Domain restriction

$\rhd$ – Range restriction

2.0.0.18. Definition

function 61 rightassoc $(\_ \lhd \_)$

$$\_ \lhd \_[X, Y] == \lambda\, S : \mathbb{P}\, X;\ R : X \leftrightarrow Y \bullet \{p : R \mid p.1 \in S\}$$

function 60 leftassoc $(\_ \rhd \_)$

$$\_ \rhd \_[X, Y] == \lambda\, R : X \leftrightarrow Y;\ T : \mathbb{P}\, Y \bullet \{p : R \mid p.2 \in T\}$$

2.0.0.19. Name $\lhd$ – Domain subtraction
$\rhd$ – Range subtraction

2.0.0.20. Definition

function 61 rightassoc $(\_ \lhd \_)$

$$\_ \lhd \_[X, Y] == \lambda S : \mathbb{P} X;\ R : X \leftrightarrow Y \bullet \{p : R \mid p.1 \notin S\}$$

function 60 leftassoc $(\_ \rhd \_)$

$$\_ \rhd \_[X, Y] == \lambda R : X \leftrightarrow Y;\ T : \mathbb{P} Y \bullet \{p : R \mid p.2 \notin T\}$$

2.0.0.21. Name $\_^{\sim}$ – Relational inversion

2.0.0.22. Definition

function$(_^{\sim})$

$$\_^{\sim}[X, Y] == \lambda R : X \leftrightarrow Y \bullet \{p : R \bullet p.2 \mapsto p.1\}$$

2.0.0.23. Name $\_ (\!| \_ |\!)$ – Relational image

2.0.0.24. Definition

$$\text{function}(\_\llparenthesis\,\_\,\rrparenthesis)$$

$$\_\llparenthesis\,\_\,\rrparenthesis[X, Y] == \lambda\, R : X \leftrightarrow Y;\ S : \mathbb{P}\, X \bullet \{p : R \mid p.1 \in S \bullet p.2\}$$

2.0.0.25. Name Upper Bound

2.0.0.26. Definition

$$\mathit{upperBound}[X, Y] == \lambda\, R : X \leftrightarrow Y \bullet \lambda\, S : \mathbb{P}\, X \bullet \{y : Y \mid S \times \{y\} \subseteq R\}$$

2.0.0.27. Description The upper bound $\mathit{upperBound}\ R\ S$ of a set S through a relation R is the set of all objects y to which R relates every member x of S.

2.0.0.28. Name

$\fatsemi$ – Relational composition

$\circ$ – Functional composition

2.0.0.29. Definition

$$\text{function}\ 40\ \text{leftassoc}\ (\_\fatsemi\_)$$

$$\_\fatsemi\_[X, Y, Z] == \lambda\, Q : X \leftrightarrow Y;\ R : Y \leftrightarrow Z \bullet \{q : Q;\ r : R \mid q.2 = r.1 \bullet q.1 \mapsto r.$$

function 40 leftassoc $(\_ \circ \_)$

$$\_ \circ \_[X, Y, Z] == \lambda\, R : Y \leftrightarrow Z;\ Q : X \leftrightarrow Y \bullet \{q : Q;\ r : R \mid q.2 = r.1 \bullet q.1 \mapsto r$$

2.0.0.30. Name *disjoint* – Disjointness
partition – Partitions

2.0.0.31. Definition

relation$(\mathit{disjoint}\ _)$

$$\mathit{disjoint}\ \_[I, X] == \{S : I \leftrightarrow \mathbb{P}\, X \mid \forall\, p, q : S \mid p \neq q \bullet p.2 \cap q.2 = \varnothing\}$$

relation$(_\ \mathit{partition}\ _)$

$$\_\ \mathit{partition}\ \_[I, X] == \{S : I \leftrightarrow \mathbb{P}\, X;\ T : \mathbb{P}\, X \mid \mathit{disjoint}\ S \land \bigcup(\mathit{ran}\ S) = T\}$$

2.0.0.32. Name $\oplus$ – Overriding

2.0.0.33. Definition

function 50 leftassoc $(\_ \oplus \_)$

$$\_ \oplus \_[X, Y] == \lambda\, Q, R : X \leftrightarrow Y \bullet (\mathit{dom}\ \ R \lhd Q) \cup R$$

2.0.0.34. Orders

2.0.0.35. Name

transitive	–	Transitive relations
antisymmetric	–	Antisymmetric relations
reflexive	–	Reflexive relations
irreflexive	–	Irreflexive relations
preOrder	–	Preorders
order	–	Partial orders
reflexiveOrder	–	Reflexive partial orders
irreflexiveOrder	–	Irreflexive partial orders
totalOrder	–	Total orders
reflexiveTotalOrder	–	Reflexive total orders
irreflexiveTotalOrder	–	Irreflexive total orders
reflexiveChain	–	Reflexive chains
irreflexiveChain	–	Irreflexive chains

2.0.0.36. Definition

$$\mathrm{generic}(\mathit{transitive}\ \_)$$

$$\mathit{transitive}\ X == \{R : X \leftrightarrow X \mid R \mathbin{;} R \subseteq R\}$$

$$\mathrm{generic}(\mathit{antisymmetric}\ \_)$$

$$\mathit{antisymmetric}\ X == \{R : X \leftrightarrow X \mid R \cap R^{\sim} \subseteq \mathit{id}\ X\}$$

$\mathrm{generic}(\mathit{reflexive}\ \_)$

$\mathit{reflexive}\ X == \{R : X \leftrightarrow X \mid \mathit{id}\ X \subseteq R\}$

$\mathrm{generic}(\mathit{irreflexive}\ \_)$

$\mathit{irreflexive}\ X == \{R : X \leftrightarrow X \mid R \cap \mathit{id}\ X = \varnothing\}$

$\mathrm{generic}(\mathit{preOrder}\ \_)$

$\mathit{preOrder}\ X == \mathit{transitive}\ X \cap \mathit{reflexive}\ X$

$\mathrm{generic}(\mathit{order}\ \_)$

$\mathit{order}\ X == \mathit{transitive}\ X \cap \mathit{antisymmetric}\ X$

$\mathrm{generic}(\mathit{reflexiveOrder}\ \_)$

$\mathit{reflexiveOrder}\ X == \mathit{reflexive}\ X \cap \mathit{order}\ X$

$\mathrm{generic}(\mathit{irreflexiveOrder}\ \_)$

$$irreflexiveOrder\ X == irreflexive\ X \cap order\ X$$

$$\mathrm{generic}(totalOrder\ \_)$$

$$totalOrder\ X == \{R : order\ X \mid R \cup R^{\sim} \cup id\ X = X \times X\}$$

$$\mathrm{generic}(reflexiveTotalOrder\ \_)$$

$$reflexiveTotalOrder\ X == reflexive\ X \cap totalOrder\ X$$

$$\mathrm{generic}(irreflexiveTotalOrder\ \_)$$

$$irreflexiveTotalOrder\ X == irreflexive\ X \cap totalOrder\ X$$

$$\begin{aligned}&reflexiveChain[X] == \lambda\, R : X \leftrightarrow X \bullet\\&\qquad \{S : \mathbb{P}\, X \mid S \lhd R \rhd S \in reflexiveTotalOrder\ S\}\end{aligned}$$

$$\begin{aligned}&irreflexiveChain[X] == \lambda\, R : X \leftrightarrow X \bullet\\&\qquad \{S : \mathbb{P}\, X \mid S \lhd R \rhd S \in irreflexiveTotalOrder\ S\}\end{aligned}$$

2.0.0.37. Name

$\_^{+}$	–	Transitive closure
$\_^{*}$	–	Reflexive-transitive closure
do	–	Functional closure

2.0.0.38. Definition

function($\_^{+}$)

$$\_^{+}[X] == \lambda R : X \leftrightarrow X \bullet \bigcap\{Q : X \leftrightarrow X \mid R \subseteq Q \land R \mathbin{;} Q \subseteq Q\}$$

function($\_^{*}$)

$$\_^{*}[X] == \lambda R : X \leftrightarrow X \bullet id\ X \cup R^{+}$$

$$do[X] == \lambda R : X \leftrightarrow X \bullet \bigcap\{Q : X \leftrightarrow X \mid id(X \setminus dom\ R) \subseteq Q \land R \mathbin{;} Q \subseteq Q\}$$

3. Functions

3.0.0.39. Name

$\nrightarrow$	–	Partial functions
$\rightarrow$	–	Total functions
$\rightarrowtail\!\!\!\!\!+$	–	Partial injections
$\rightarrowtail$	–	Total injections
$\twoheadrightarrow\!\!\!\!\!+$	–	Partial surjections
$\twoheadrightarrow$	–	Total surjections
$\rightarrowtail\!\!\!\!\twoheadrightarrow$	–	Bijections

3.0.0.40. Definition

generic 5 rightassoc $(\_ \pfun \_)$

$$X \pfun Y == \{f : X \rel Y \mid \forall p, q : f \mid p.1 = q.1 \bullet p = q\}$$

generic 5 rightassoc $(\_ \fun \_)$

$$X \fun Y == \{f : X \pfun Y \mid dom\ f = X\}$$

generic 5 rightassoc $(\_ \pinj \_)$

$$X \pinj Y == \{f : X \rel Y \mid \forall p, q : f \bullet p.1 = q.1 \Leftrightarrow p.2 = q.2\}$$

generic 5 rightassoc $(\_ \inj \_)$

$$X \inj Y == \{f : X \pinj Y \mid dom\ f = X\}$$

generic 5 rightassoc $(\_ \psurj \_)$

$$X \psurj Y == \{f : X \pfun Y \mid ran\, f = Y\}$$

generic 5 rightassoc $(\_ \surj \_)$

$$X \surj Y == \{f : X \fun Y \mid ran\, f = Y\}$$

generic 5 rightassoc $(\_ \bij \_)$

$$X \bij Y == \{f : X \inj Y \mid ran\, f = Y\}$$

3.0.0.41. Name

$\twoheadrightarrow\!\!\!\!\!\!\!\!\;\;\;\shortmid\!\shortmid$ — Finite functions

$\rightarrowtail\!\!\!\!\!\!\!\shortmid\!\shortmid\!\!\!\rightarrow$ — Finite injections

3.0.0.42. Definition

generic 5 rightassoc $(\_ \mathrel{+\!\!\!+\!\!\!\!\rightarrow} \_)$

$$X \mathrel{+\!\!\!+\!\!\!\!\rightarrow} Y == \{f : X \mathrel{+\!\!\!\!\rightarrow} Y \mid \mathit{finite}\ f\}$$

generic 5 rightassoc $(\_ \mathrel{\rightarrowtail\!\!\!\!\!\!\!+\!\!\!+\!\!\!\!\rightarrow} \_)$

$$X \mathrel{\rightarrowtail\!\!\!\!\!\!\!+\!\!\!+\!\!\!\!\rightarrow} Y == \{f : X \mathrel{\rightarrowtail\!\!\!\!\!\!\!+\!\!\!\!\rightarrow} Y \mid \mathit{finite}\ f\}$$

IT 5-Jan-2002