section relation_toolkit

/Reference manual/Standard toolkit

1. More notations for relations

section *relation_toolkit* parents *set_toolkit*

1.1. First component projection

$$
\begin{array}{l}
[X, Y] \\
\hline
\mathit{first} : X \times Y \rightarrow X \\
\hline
\forall x : X;\ y : Y \bullet \mathit{first}\ (x, y) = x
\end{array}
$$

For any ordered pair (x, y), $\mathit{first}\ (x, y)$ is the x component of the pair.

1.2. Second component projection

$$
\begin{array}{l}
[X, Y] \\
\hline
\mathit{second} : X \times Y \rightarrow Y \\
\hline
\forall x : X;\ y : Y \bullet \mathit{second}\ (x, y) = y
\end{array}
$$

For any ordered pair (x, y), *second* (x, y) is the y component of the pair.

1.3. Maplet

$$\text{function } 10 \text{ leftassoc}(\_ \mapsto \_)$$

$$\begin{array}{l} [X, Y] \\ \hline \_ \mapsto \_ : X \times Y \rightarrow X \times Y \\ \hline \forall x : X;\ y : Y \bullet x \mapsto y = (x, y) \end{array}$$

The maplet forms an ordered pair from two values; $x \mapsto y$ is just another notation for (x, y).

1.4. Domain

$$\begin{array}{l} [X, Y] \\ \hline \mathit{dom} : (X \leftrightarrow Y) \rightarrow \mathbb{P}\, X \\ \hline \forall r : X \leftrightarrow Y \bullet \mathit{dom}\, r = \{\, p : r \bullet p.1 \,\} \end{array}$$

The domain of a relation r is the set of first components of the ordered pairs in r.

1.5. Range

$$
\begin{array}{|l}
\hline
[X, Y] \\
ran : (X \leftrightarrow Y) \rightarrow \mathbb{P}\, Y \\
\hline
\forall\, r : X \leftrightarrow Y \bullet ran\, r = \{\, p : r \bullet p.2 \,\}
\end{array}
$$

The range of a relation r is the set of second components of the ordered pairs in r.

1.6. Identity relation

$$\text{generic}(id\, \_)$$

$$id\, X == \{\, x : X \bullet x \mapsto x \,\}$$

The identity relation on a set X is the relation that relates every member of X to itself.

1.7. Relational composition

$$\text{function 40 leftassoc}(\_ \mathbin{⨾} \_)$$

$$
\begin{array}{|l}
\hline
[X, Y, Z] \\
\_ \mathbin{⨾} \_ : (X \leftrightarrow Y) \times (Y \leftrightarrow Z) \rightarrow (X \leftrightarrow Z) \\
\hline
\forall r : X \leftrightarrow Y;\ s : Y \leftrightarrow Z \bullet r \mathbin{⨾} s = \{\ p : r;\ q : s \mid p.2 = q.1 \bullet p.1 \mapsto q.2\ \}
\end{array}
$$

The relational composition of a relation $r : X \leftrightarrow Y$ and $s : Y \leftrightarrow Z$ is a relation of type $X \leftrightarrow Z$ formed by taking all the pairs p of r and q of s, where the second component of p is equal to the first component of q, and relating the first component of p with the second component of q.

1.8. Functional composition

function 40 leftassoc($\_ \circ \_$)

$$
\begin{array}{|l}
\hline
[X, Y, Z] \\
\_ \circ \_ : (Y \leftrightarrow Z) \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Z) \\
\hline
\forall r : X \leftrightarrow Y;\ s : Y \leftrightarrow Z \bullet s \circ r = r \mathbin{⨾} s
\end{array}
$$

The functional composition of s and r is the same as the relational composition of r and s.

1.9. Domain restriction

function 65 rightassoc($\_ \lhd \_$)

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \lhd \_ : \mathbb{P}\, X \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Y) \\
\hline
\forall a : \mathbb{P}\, X;\ r : X \leftrightarrow Y \bullet a \lhd r = \{\ p : r \mid p.1 \in a\ \}
\end{array}
$$

The domain restriction of a relation $r : X \leftrightarrow Y$ by a set $a : \mathbb{P}\, X$ is the set of pairs in r whose first components are in a.

1.10. Range restriction

function 60 leftassoc$(_ \rhd _)$

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \rhd \_ : (X \leftrightarrow Y) \times \mathbb{P}\, Y \rightarrow (X \leftrightarrow Y) \\
\hline
\forall r : X \leftrightarrow Y;\ b : \mathbb{P}\, Y \bullet r \rhd b = \{\ p : r \mid p.2 \in b\ \}
\end{array}
$$

The range restriction of a relation $r : X \leftrightarrow Y$ by a set $b : \mathbb{P}\, Y$ is the set of pairs in r whose second components are in b.

1.11. Domain subtraction

function 65 rightassoc$(_ \ntriangleleft _)$

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \lhd \_ : \mathbb{P}\, X \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Y) \\
\hline
\forall\, a : \mathbb{P}\, X;\ r : X \leftrightarrow Y \bullet a \lhd r = \{\, p : r \mid p.1 \notin a \,\}
\end{array}
$$

The domain subtraction of a relation $r : X \leftrightarrow Y$ by a set $a : \mathbb{P}\, X$ is the set of pairs in r whose first components are not in a.

1.12. Range subtraction

function 60 leftassoc($\_ \rhd \_$)

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \rhd \_ : (X \leftrightarrow Y) \times \mathbb{P}\, Y \rightarrow (X \leftrightarrow Y) \\
\hline
\forall\, r : X \leftrightarrow Y;\ b : \mathbb{P}\, Y \bullet r \rhd b = \{\, p : r \mid p.2 \notin b \,\}
\end{array}
$$

The range subtraction of a relation $r : X \leftrightarrow Y$ by a set $b : \mathbb{P}\, Y$ is the set of pairs in r whose second components are not in b.

1.13. Relational inversion

function($\_^{\sim}$)

$$
\begin{array}{l}
[X, Y] \\
\hline
\_^{\sim} : (X \leftrightarrow Y) \rightarrow (Y \leftrightarrow X) \\
\hline
\forall r : X \leftrightarrow Y \bullet r^{\sim} = \{ p : r \bullet p.2 \mapsto p.1 \}
\end{array}
$$

The inverse of a relation is the relation obtained by reversing every ordered pair in the relation.

1.14. Relational image

$$\mathrm{function}(\_ (\!| \_ |\!))$$

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ (\!| \_ |\!) : (X \leftrightarrow Y) \times \mathbb{P}\, X \rightarrow \mathbb{P}\, Y \\
\hline
\forall r : X \leftrightarrow Y;\ a : \mathbb{P}\, X \bullet r (\!| a |\!) = \{ p : r \mid p.1 \in a \bullet p.2 \}
\end{array}
$$

The relational image of a set $a : \mathbb{P}\, X$ through a relation $r : X \leftrightarrow Y$ is the set of values of type Y that are related under r to a value in a.

1.15. Overriding

$$\mathrm{function}\ 50\ \mathrm{leftassoc}(\_ \oplus \_)$$

$$
\begin{array}{l}
[X, Y] \\
\_ \oplus \_ : (X \leftrightarrow Y) \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Y) \\
\hline
\forall r, s : X \leftrightarrow Y \bullet r \oplus s = ((\mathit{dom}\, s) \lhd r) \cup s
\end{array}
$$

If r and s are both relations between X and Y, the overriding of r by s is the whole of s together with those members of r that have no first components that are in the domain of s.

1.16. Transitive closure

function($\_^{+}$)

$$
\begin{array}{l}
[X] \\
\_^{+} : (X \leftrightarrow X) \rightarrow (X \leftrightarrow X) \\
\hline
\forall r : X \leftrightarrow X \bullet r^{+} = \bigcap \{\, s : X \leftrightarrow X \mid r \subseteq s \land r \mathbin{;} s \subseteq s \,\}
\end{array}
$$

The transitive closure of a relation $r : X \leftrightarrow X$ is the smallest set that contains r and is closed under the action of composing r with its members.

1.17. Reflexive transitive closure

function($\_^{*}$)

$$[X]$$
$$\_^{*} : (X \leftrightarrow X) \rightarrow (X \leftrightarrow X)$$
$$\forall r : X \leftrightarrow X \bullet r^{*} = r^{+} \cup id\ X$$

The reflexive transitive closure of a relation $r : X \leftrightarrow X$ is the relation formed by extending the transitive closure of r by the identity relation on X.

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