section sequence_toolkit

1. Sequences

section *sequence_toolkit* parents *function_toolkit*, *number_toolkit*

1.1. Number range

function 20 leftassoc($\_ \mathinner{\ldotp\ldotp} \_$)

$$
\begin{array}{|l}
\_ \mathinner{\ldotp\ldotp} \_ : \mathbb{A} \times \mathbb{A} \leftrightarrow \mathbb{P}\,\mathbb{A} \\
\hline
(\mathbb{Z} \times \mathbb{Z}) \lhd (\_ \mathinner{\ldotp\ldotp} \_) \in \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{P}\,\mathbb{Z} \\
\forall\, i, j : \mathbb{Z} \bullet i \mathinner{\ldotp\ldotp} j = \{\, k : \mathbb{Z} \mid i \leq k \leq j \,\}
\end{array}
$$

The number range from i to j is the set of all integers greater than or equal to i, which are also less than or equal to j.

1.2. Iteration

$$
\begin{array}{|l}
\hline
[X] \\
\mathit{iter} : \mathbb{Z} \rightarrow (X \leftrightarrow X) \rightarrow (X \leftrightarrow X) \\
\hline
\forall r : X \leftrightarrow X \bullet \mathit{iter}\ 0\ r = \mathit{id}\ X \\
\forall r : X \leftrightarrow X;\ n : \mathbb{N} \bullet \mathit{iter}\ (n + 1)\ r = r \fatsemi (\mathit{iter}\ n\ r) \\
\forall r : X \leftrightarrow X;\ n : \mathbb{N} \bullet \mathit{iter}\ (\text{-}n)\ r = \mathit{iter}\ n\ (r^{\sim})
\end{array}
$$

iter is the iteration function for a relation. The iteration of a relation $r : X \leftrightarrow X$ for zero times is the identity relation on X. The iteration of a relation $r : X \leftrightarrow X$ for $n + 1$ times is the composition of the relation with its iteration n times. The iteration of a relation $r : X \leftrightarrow X$ for -n times is the iteration for n times of the inverse of the relation.

function($\_ ^{\_}$)

$$
\begin{array}{|l}
\hline
[X] \\
\_ ^{\_} : (X \leftrightarrow X) \times \mathbb{Z} \rightarrow (X \leftrightarrow X) \\
\hline
\forall r : X \leftrightarrow X;\ n : \mathbb{N} \bullet r^{\ n} = \mathit{iter}\ n\ r
\end{array}
$$

$\mathit{iter}\ n\ r$ may be written as r^n.

1.3. Number of members of a set

function($\#\_$)

$$
\begin{array}{l}
[X] \\
\hline
\#\_ : \mathbb{F}\, X \rightarrow \mathbb{N} \\
\hline
\forall\, a : \mathbb{F}\, X \bullet \#a = (\mu\, n : \mathbb{N} \mid (\exists\, f : 1 \mathrel{..} n \rightarrowtail a \bullet \mathrm{ran}\, f = a))
\end{array}
$$

The number of members of a finite set is the upper limit of the number range starting at 1 that can be put into bijection with the set.

1.4. Minimum

function(min _)

$$
\begin{array}{l}
min\ \_ : \mathbb{P}\, \mathbb{A} \nrightarrow \mathbb{A} \\
\hline
\mathbb{P}\, \mathbb{Z} \lhd (min\ \_) = \{\ a : \mathbb{P}\, \mathbb{Z};\ m : \mathbb{Z} \mid m \in a \land (\forall\, n : a \bullet m \leq n) \bullet a \mapsto m\ \}
\end{array}
$$

If a set of integers has a member that is less than or equal to all members of that set, that member is its minimum.

1.5. Maximum

function(max _)

$$\begin{array}{|l} max\ \_ : \mathbb{P}\,\mathbb{A} \nrightarrow \mathbb{A} \\ \hline \mathbb{P}\,\mathbb{Z} \lhd (max\ \_) = \{\ a : \mathbb{P}\,\mathbb{Z};\ m : \mathbb{Z} \mid m \in a \land (\forall n : a \bullet n \leq m) \bullet a \mapsto m\ \} \end{array}$$

If a set of integers has a member that is greater than or equal to all members of that set, that member is its maximum.

1.6. Finite sequences

$$\text{generic}(seq\ \_)$$

$$seq\ X == \{\ f : \mathbb{N} \mathrel{\nrightarrow\!\!\!\!\!\!\nrightarrow} X \mid dom\ f = 1 \mathinner{..} \# f\ \}$$

A finite sequence is a finite indexed set of values of the same type, whose domain is a contiguous set of positive integers starting at 1.

$seq\ X$ is the set of all finite sequences of values of X, that is, of finite functions from the set $1 \mathinner{..} n$, for some n, to elements of X.

1.7. Non-empty finite sequences

$$seq_1\ X == seq\ X \setminus \{\varnothing\}$$

$seq_1\ X$ is the set of all non-empty finite sequences of values of X.

1.8. Injective sequences

$$\mathrm{generic}(\mathit{iseq}\,\_)$$

$$\mathit{iseq}\,X == \mathit{seq}\,X \cap (\mathbb{N} \rightarrowtail\!\!\!\!\!\twoheadrightarrow X)$$

$\mathit{iseq}\,X$ is the set of all injective finite sequences of values of X, that is, of finite sequences over X that are also injections.

1.9. Sequence brackets

$$\mathrm{function}(\langle ,, \rangle)$$

$$\langle ,, \rangle[X] == \lambda\, s : \mathit{seq}\,X \bullet s$$

The brackets $\langle$ and $\rangle$ can be used for enumerated sequences.

1.10. Concatenation

$$\mathrm{function}\,30\,\mathrm{leftassoc}(\_ \frown \_)$$

$$
\begin{array}{l}
[X] \\
\_ \frown \_ : seq\ X \times seq\ X \rightarrow seq\ X \\
\hline
\forall\, s, t : seq\ X \bullet s \frown t = s \cup \{\ n : dom\ t \bullet n + \#s \mapsto t\ \ n\ \}
\end{array}
$$

Concatenation is a function of a pair of finite sequences of values of the same type whose result is a sequence that begins with all elements of the first sequence and continues with all elements of the second sequence.

1.11. Reverse

$$
\begin{array}{l}
[X] \\
rev : seq\ X \rightarrow seq\ X \\
\hline
\forall\, s : seq\ X \bullet rev\ \ s = \lambda\, n : dom\ s \bullet s(\#s - n + 1)
\end{array}
$$

The reverse of a sequence is the sequence obtained by taking its elements in the opposite order.

1.12. Head of a sequence

$$
\begin{array}{l}
[X] \\
head : seq_1\ X \rightarrow X \\
\hline
\forall\, s : seq_1\ X \bullet head\ \ s = s\ 1
\end{array}
$$

If s is a non-empty sequence of values, then $head\ s$ is the value that is first in the sequence.

1.13. Last of a sequence

$$
\begin{array}{l}
[X] \\
\hline
last : seq_1\, X \rightarrow X \\
\hline
\forall\, s : seq_1\, X \bullet last\ s = s(\# s)
\end{array}
$$

If s is a non-empty sequence of values, then $last\ s$ is the value that is last in the sequence.

1.14. Tail of a sequence

$$
\begin{array}{l}
[X] \\
\hline
tail : seq_1\, X \rightarrow seq\, X \\
\hline
\forall\, s : seq_1\, X \bullet tail\ s = \lambda\, n : 1 \mathrel{..} (\# s - 1) \bullet s(n + 1)
\end{array}
$$

If s is a non-empty sequence of values, then $tail\ s$ is the sequence of values that is obtained from s by discarding the first element and renumbering the remainder.

1.15. Front of a sequence

$$
\begin{array}{l}
[X] \\
\hline
\mathit{front} : \mathrm{seq}_1 X \rightarrow \mathrm{seq}\, X \\
\hline
\forall s : \mathrm{seq}_1 X \bullet \mathit{front}\ s = \{\#s\} \ntriangleleft s
\end{array}
$$

If s is a non-empty sequence of values, then $\mathit{front}\ s$ is the sequence of values that is obtained from s by discarding the last element.

1.16. Squashing

$$
\begin{array}{l}
[X] \\
\hline
\mathit{squash} : (\mathbb{Z} \mathrel{+\!\!\!\!\!\twoheadrightarrow} X) \rightarrow \mathrm{seq}\, X \\
\hline
\forall f : \mathbb{Z} \mathrel{+\!\!\!\!\!\twoheadrightarrow} X \bullet \mathit{squash}\ f = \{\ p : f \bullet \#\{\ i : \mathrm{dom}\, f \mid i \leq p.1\ \} \mapsto p.2\ \}
\end{array}
$$

squash takes a finite function $f : \mathbb{Z} \mathrel{+\!\!\!\!\!\twoheadrightarrow} X$ and renumbers its domain to produce a finite sequence.

1.17. Extraction

function 45 rightassoc$(_ \upharpoonleft _)$

$$
\begin{array}{l}
[X] \\
\hline
\_ \upharpoonleft \_ : \mathbb{P}\,\mathbb{Z} \times \mathit{seq}\ X \rightarrow \mathit{seq}\ X \\
\hline
\forall a : \mathbb{P}\,\mathbb{Z};\ s : \mathit{seq}\ X \bullet a \upharpoonleft s = \mathit{squash}(a \lhd s)
\end{array}
$$

The extraction of a set a of indices from a sequence is the sequence obtained from the original by discarding any indices that are not in the set a, then renumbering the remainder.

1.18. Filtering

function 40 leftassoc$(_ \upharpoonright _)$

$$
\begin{array}{l}
[X] \\
\hline
\_ \upharpoonright \_ : \mathit{seq}\ X \times \mathbb{P}\,X \rightarrow \mathit{seq}\ X \\
\hline
\forall s : \mathit{seq}\ X;\ a : \mathbb{P}\,X \bullet s \upharpoonright a = \mathit{squash}(s \rhd a)
\end{array}
$$

The filter of a sequence by a set a is the sequence obtained from the original by discarding any members that are not in the set a, then renumbering the remainder.

1.19. Prefix relation

relation$(_ \ \mathit{prefix}\ _)$

$$
\begin{array}{l}
[X] \\
\hline
\_\,\mathit{prefix}\,\_ : \mathit{seq}\ X \leftrightarrow \mathit{seq}\ X \\
\hline
\forall\, s, t : \mathit{seq}\ X \bullet s\ \mathit{prefix}\ t \Leftrightarrow s \subseteq t
\end{array}
$$

A sequence s is a prefix of another sequence t if it forms the front portion of t.

1.20. Suffix relation

relation($\_\,\mathit{suffix}\,\_$)

$$
\begin{array}{l}
[X] \\
\hline
\_\,\mathit{suffix}\,\_ : \mathit{seq}\ X \leftrightarrow \mathit{seq}\ X \\
\hline
\forall\, s, t : \mathit{seq}\ X \bullet s\ \mathit{suffix}\ t \Leftrightarrow (\exists\, u : \mathit{seq}\ X \bullet u \frown s = t)
\end{array}
$$

A sequence s is a suffix of another sequence t if it forms the end portion of t.

1.21. Infix relation

relation($\_\,\mathrm{infix}\,\_$)

$$\begin{array}{l} [X] \\ \_\,\mathrm{infix}\,\_ : \mathit{seq}\ X \leftrightarrow \mathit{seq}\ X \\ \hline \forall\, s, t : \mathit{seq}\ X \bullet s\ \mathrm{infix}\ t \Leftrightarrow (\exists\, u, v : \mathit{seq}\ X \bullet u \frown s \frown v = t) \end{array}$$

A sequence s is an infix of another sequence t if it forms a mid portion of t.

1.22. Distributed concatenation

$$\begin{array}{l} [X] \\ \frown/ : \mathit{seq}\ \mathit{seq}\ X \rightarrow \mathit{seq}\ X \\ \hline \frown/\langle\rangle = \langle\rangle \\ \forall\, s : \mathit{seq}\ X \bullet \frown/\langle s\rangle = s \\ \forall\, q, r : \mathit{seq}\ \mathit{seq}\ X \bullet \frown/(q \frown r) = (\frown/\, q) \frown (\frown/\, r) \end{array}$$

The distributed concatenation of a sequence t of sequences of values of type X is the sequence of values of type X that is obtained by concatenating the members of t in order.

IT 22-Jan-2002